Cyclic (co)homology

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(Co)homology Theory

- Manifolds: de Rham, singular and Cech cohomology.

Assembling local data to extract global information.

- Group homology, Lie algebra homology
  and their relation with Hopf cyclic cohomology.

Cyclic cohomology

- Cyclic cohomology (of algebras) was discovered by Alain Connes no later than 1981.
- One of Connes main motivations to introduce cyclic cohomology theory came from index theory on foliated spaces.

Cocycle

- $$\text{cyclic cocycle} = \text{cohomology class}$$
- cyclic cocycles have:
  topological information
  algebraic information
  geometric information

Cyclic Cocycle For Algebras

Definition

- $A =$ algebra.
- cyclic n-cocycle

$$\varphi : A \otimes \cdots \otimes A \longrightarrow \mathbb{C}$$

- $$b\varphi = 0, \quad \lambda\varphi = \varphi$$

- $$(\lambda\varphi)(a_0, \cdots, a_n) = (-1)^n \varphi(a_n, a_0, \cdots, a_{n-1}).$$

- $(b\varphi)(a_0, \cdots, a_n) = \sum_{i=0}(-1)^i \varphi(a_0, \cdots, a_i a_{i+1} \cdots, a_n) + (-1)^{n+1}\varphi(a_n a_0, a_1, \cdots, a_{n-1})$

Example, Even cocycles= Trace Maps

- $A = M_n(\mathbb{C})$.
- cyclic cocycle

$$\varphi_{2n} : A^{\otimes(2n+1)} \longrightarrow \mathbb{C}$$

$$\varphi_{2n}(a_0, \cdots, a_{2n}) = Tr(a_0 a_1 \cdots a_{2n})$$

Example: Cyclic Cocycle

- M = closed (i.e. compact without boundary), smooth, oriented, n-manifold.
- $A = C^{\infty}(M)$: smooth complex valued functions.
- $f_0, \cdots, f_n \in A$.
- $$\varphi : A \otimes \cdots \otimes A \longrightarrow \mathbb{C}$$

$$\varphi(f_0, \cdots, f_n) = \int_M f_0 df_1 \wedge \cdots \wedge df_n$$

Previous Example

Properties of φ:

- φ continuous.
- φ Hochschild cocycle.
- φ cyclic

Continuity of φ

- φ continuous in Frechet space topology of A:

$$f_n \longrightarrow f \Longleftrightarrow \partial_\alpha f_n \longrightarrow \partial_\alpha f$$

Uniformly in a coordinate system.

φ is Hochschild Cocycle

- $$b\varphi = 0$$

- $$(b\varphi)(f_0, \cdots, f_{n+1}) =$$
$$\sum (-1)^i \int_M f_0 df_1 \cdots d(f_i f_{i+1}) \cdots df_{n+1} +$$
$$(-1)^{n+1} \int_M f_{n+1} f_0 df_1 \cdots df_n =$$
$$0.$$

- Leibnitz rule, $d(fg) = f'g + fg'$.
- $df \wedge dg = -dg \wedge df$.

Cyclicity of φ

- φ is cyclic.

$$\varphi(f_n, f_0, \cdots, f_{n-1}) = (-1)^n \varphi(f_0, \cdots, f_n)$$

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- Since

$$\int_M (f_n df_0 \cdots df_{n-1} - (-1)^n f_0 df_1 \cdots df_n) = \int_M d(f_n f_0 df_1 \cdots df_{n-1}).$$

- Stokes formula

$$\int_M d\omega = 0$$

$\omega = (n-1)$-form.

Example

$\textit{cyclic} \quad \textit{cocycle} \rightsquigarrow \textit{algebraic} \quad \textit{information}$

0-Cocycles = Trace Maps

- A = Any algebra.
- All 0-cocycles:

$$\varphi : A \longrightarrow \mathbb{C}$$

$$\varphi(ab) = \varphi(ba)$$

$$\varphi = \text{Trace}$$

Cyclic 1-Cocycle

- All Cyclic 1-Cocycles:

$$\varphi : A \otimes A \longrightarrow \mathbb{C}$$

- Satisfying following two conditions,

$$\varphi(ab, c) - \varphi(a, bc) + \varphi(ca, b) = 0$$

$$\varphi(b, a) = -\varphi(a, b)$$

Example

cyclic cocycle $\rightsquigarrow$ *topological information*

Winding Number And Cyclic 1-Cocycles

- $A = C^\infty(S^1)$.
- $$\varphi(f_0, f_1) = \frac{1}{2\pi i}\int_{S^1} f_0 df_1$$
- φ is 1-cyclic cocycle. (Already Shown)
- f invertible:

$$\varphi(f^{-1}, f) = \frac{1}{2\pi i}\int_{S^1} f^{-1} df = W(f, 0)$$

Winding Number

Generalization Of Previous Example

- $\delta : A \longrightarrow A$ derivation; $\delta(ab) = \delta(a)b + a\delta(b)$.
- $\tau : A \longrightarrow \mathbb{C}$ invariant trace:

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- cyclic 1-cocycle

$$\varphi(a_0, a_1) = \tau(a_0\delta(a_1))$$

- Generalizes

$$\varphi(f_0, f_1) = \int_{S^1} f_0 df_1$$

- $A = C^\infty(S^1)$, $\delta = d$, $\tau = \int$.

Cyclic 2-cocycles Of Previous Example

- δ_1, δ_2 derivations
  τ-invariant
  $\delta_1\delta_2 = \delta_2\delta_1$
- cyclic 2-cocycle

$$\varphi(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_1(a_1) - \delta_2(a_1)\delta_1(a_2))$$

Application, Noncommutative Torus

- $A = A_\theta =$ noncommutative torus.
  $\tau =$ standard trace.
- $$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \delta_2(U) = 0, \quad \delta_2(V) = V$$

Example

- $A = C^{\infty}(M)$.
- $V \subseteq M =$ closed p-dimensional oriented submanifold.
- $$\varphi(f_0, \cdots, f_p) = \int_V f_0 df_1 \cdots df_p$$
- φ is a cyclic p-cocycle.

More Examples: de Rham Homology

- A p-dimensional current C on M is a continuous map

$$\Phi : \Omega^p M \longrightarrow \mathbb{C}$$

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$$\cdots \xrightarrow{d} C^1(M) \xrightarrow{d} C^0(M)$$

-

$$d\Phi(\omega) = \Phi(d\omega)$$

- $d^2 = 0.$
- $H_d R^*(M) =$ de Rham homology

De Rham homology and cyclic cohomology

- $A = C^\infty(M)$.
- Φ is p-dimensional current on M.
- The $(p+1)$-linear functional

$$\varphi_\Phi(f_0, \cdots, f_p) = \Phi(f_0 df_1 \cdots df_p)$$

- φ_Φ Hochschild cocycle.
- Φ closed $\rightsquigarrow$ φ_Φ cycllic cocycle.
  Closed: $\Phi(d\omega) = 0, \omega \in \Omega^{p-1}(M)$.
- $\{$ closed de Rham p-currents on $M\} \rightsquigarrow \{$cyclic p-cocycles on $C^\infty(M)\}$

Duality: De Rham homology and cyclic homology

- { closed de Rham p-currents on M} $\hookleftarrow$ {cyclic p-cocycles on $C^\infty(M)$}
- Exercise!

Theorem

de Rham homology $\leftrightsquigarrow$ *cyclic cohomology* $C^\infty(M)$

Topological cyclic homology(Connes-1985)

- $A = C^{\infty}(M)$
- Noncommutative Torus
- Other computations of cyclic cohomology. (Khalkhali-Rangipour , Kustermans-Rognes-Tuset and Hadfield-Krahmer)

Connes cyclic modules(1983)

- Connes defined the notion of a cyclic object in an abelian category and its cyclic cohomology.
- conceptualizing and generalizing cyclic cohomology far beyond its original inception.
  Motivation: cyclic cohomology of algebras as a derived functor.

Cosimplicial Module

Definition

A cosimplicial module (C^n, δ_i^n, s_i^n), where, C^n, $n \geq 0$ k-modules with k-module maps $\delta_i^n : C^n \longrightarrow C^{n+1}$ cofaces, $s_i^n : C^n \longrightarrow C^{n-1}$ codegeneracies,

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$$\begin{aligned}
\delta_j^n \delta_i^{n-1} &= \delta_i^n \delta_{j-1}^{n-1} \quad \text{if} \quad i < j, \\
s_j^n s_i^{n+1} &= s_i^n s_{j+1}^{n+1} \quad \text{if} \quad i \leq j, \\
s_j^n \delta_i^{n+1} &= \begin{cases} \delta_i^n s_{j-1}^{n-1} & \text{if} \quad i < j \\ \text{id} & \text{if} \quad i = j \text{ or } i = j+1 \\ \delta_{i-1}^n s_j^{n-1} & \text{if} \quad i > j+1. \end{cases}
\end{aligned} \tag{0.1}$$

Definition

A cocyclic module $(C^n, \delta_i^n, s_i^n, \tau_n)$, where (C^n, δ_i^n, s_i^n) is cosimplicial module with $\tau_n : C^n \longrightarrow C^n$, called cocyclic map

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$$\begin{aligned} \tau_n \delta_i^n &= \delta_{i-1}^n \tau_{n-1} && 1 \leq i \leq n \\ \tau_n \delta_0^n &= \delta_n^n \\ \tau_n s_i^n &= s_{i-1}^n \tau_{n+1} && 1 \leq i \leq n \\ \tau_n s_0^n &= s_n^n \tau_{n+1}^2 \\ \tau_n^{n+1} &= \text{id}. \end{aligned} \tag{0.2}$$

Example, A Cocylic Module for Unital Algebras

Let $C^n(A) = \mathrm{Hom}_k(A^{\otimes(n+1)}, k)$.

$$\delta_i\varphi(a_0 \otimes \dots \otimes a_n) = \begin{cases} \varphi(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n) & 0 \leq i < n \\ \varphi(a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}) & i = n \end{cases}$$

$$\sigma_i\varphi(a_0 \otimes \dots \otimes a_n) = \varphi(a_0 \otimes \dots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \dots \otimes a_n), \quad 0 \leq i \leq n$$

$$\tau_n\varphi(a_0 \otimes \dots \otimes a_n) = \varphi(a_n \otimes a_0 \otimes \dots \otimes a_{n-1}).$$

Hochschild Cohomology of a Cocyclic Module, $HH^*(C)$

Definition

$C = (C^n, \delta_i^n, s_i^n) =$ Cosimplicial module in a abelian category. The Hochschild cohomology, $HH^*(C)$

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$$C^0 \xrightarrow{b} C^1 \xrightarrow{b} C^2 \xrightarrow{b} C^3 \ldots,$$

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$$C^0 \xrightarrow{b} C^1 \xrightarrow{b} C^2 \xrightarrow{b} C^3 \ldots,$$

where $b : C^{n-1} \longrightarrow C^n$ is defined by

$$b = \sum_{i=0}^{n} (-1)^i \delta_i^n.$$

- $b^2 = 0$
- $H^*(C, b) = HH^*(C)$ = Hochschild cohomology of C.

Cyclic Cohomology of a Cocyclic Module, $HC^*(C)$

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Cyclic Cohomology of a Cocyclic Module, $HC^*(C)$

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(b, B)-bicomplex $\mathcal{B}^{**}(C)$:

Cyclic Cohomology of a Cocyclic Module, $HC^*(C)$

$C = (C^n, \delta_i^n, s_i^n, \tau_n) =$ Cocyclic module in abelian category,
(b, B)-bicomplex $\mathcal{B}^{**}(C)$:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
C^2 & \xrightarrow{B} & C^1 & \xrightarrow{B} & C^0 \\
b\uparrow & & b\uparrow & & \\
C^1 & \xrightarrow{B} & C^0 & & \\
b\uparrow & & & & \\
C^0 & & & &
\end{array}
$$

where

$$B = Ns(1 - \lambda).$$

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$$\lambda_n = (-1)^n \tau_n,$$

$$N = 1 + \lambda + \lambda^2 + ... + \lambda^n.$$

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and

$$s = s_n^n \tau_{n+1} : C^{n+1} \to C^n.$$

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$$\lambda_n = (-1)^n \tau_n,$$

$$N = 1 + \lambda + \lambda^2 + \ldots + \lambda^n.$$

and

$$s = s_n^n \tau_{n+1} : C^{n+1} \to C^n.$$

$$HC^n(C) := H^n(\mathrm{Tot}\mathcal{B}^{**}(C)).$$

Periodic Cyclic Cohomology of a Cocyclic Module, $HP^*(C)$

The bicomplex, $\widehat{\mathcal{B}}^{**}(C)$

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cdots C^4 & \xrightarrow{B} & C^3 & \xrightarrow{B} & C^2 & \xrightarrow{B} & C^1 & \xrightarrow{B} & C^0 \\
b\uparrow & & b\uparrow & & b\uparrow & & b\uparrow & & \\
\cdots C^3 & \xrightarrow{B} & C^2 & \xrightarrow{B} & C^1 & \xrightarrow{B} & C^0 & & \\
b\uparrow & & b\uparrow & & b\uparrow & & & & \\
\cdots C^2 & \xrightarrow{B} & C^1 & \xrightarrow{B} & C^0 & & & & \\
b\uparrow & & b\uparrow & & & & & & \\
\cdots C^1 & \xrightarrow{B} & C^0 & & & & & & \\
b\uparrow & & & & & & & &
\end{array}
$$

END

Thanks